MS&E 319: Matching Theory

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HW#1 – Due Tuesday May 3, 2022

1. Prove that the edges of a bipartite graph with maximum degree Δ can be colored with Δ colors such that no two edges that share a vertex have the same color.

2. A square matrix $A \in \mathbb{R}^{n \times n}$ is *doubly stochastic* if the entries of the matrix are nonnegative, and the sum of entries in every row and column is equal to one. The Birkhoff-von Neumann theorem states that one can write any doubly stochastic matrix as a convex combination of permutation matrices. Prove this theorem, and furthermore show that it suffices to use a convex combination of at most $n^2 - n$ permutation matrices.

3. An *independent set* S in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. If we let P denote the convex hull of all (incidence vectors of) independent sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for P.

   (a) Give a graph G for which P is *not* equal to

   $$\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \quad \text{for all } (i,j) \in E$$
   $$x_i \geq 0 \quad \text{for all } i \in V\}$$

   (b) Show that if the graph G is bipartite then P equals

   $$\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \quad \text{for all } (i,j) \in E$$
   $$x_i \geq 0 \quad \text{for all } i \in V\}.$$

4. (Echenique, Immorlica, Vazirani 1.15) Consider an instance of the stable matching problem with n doctors and n hospitals, each with capacity 1. Assume there are an odd number, k, of stable matchings. For each doctor d, order his or her k matches (with repetitions) according to his or her preference list and do the same for every hospital h. Consider assigning every doctor to the *median* hospital in his or her list. In this problem, we will prove that this is a stable matching.

   To do so, first let $\mu_1, \ldots, \mu_l$ be any l stable matchings. For each doctor-hospital pair (d, h), let $n(d, h)$ be the number of matchings in $\{\mu_1, \ldots, \mu_l\}$ where d is matched to h. Define $x_{dh} := (1/l) \cdot n(d, h)$. **Show that x is a feasible solution to the fractional stable matching LP.** For any k with $1 \leq k \leq l$, let $\theta = (k/l) - \epsilon$ where $\epsilon > 0$ is smaller than $1/l$. Consider the stable matching μ_θ formed by "rounding" the fractional stable matching x with θ via the procedure in class.[1] **Show that μ_θ matches each doctor d to the k^{th} hospital in his or her ordered list of the l firms d is matched to in $\{\mu_1, \ldots, \mu_l\}$. Show similarly that hospital h is matched to the $(l - k + 1)^{\text{th}}$ doctor in its list.**

   Use this to show that the aforementioned "median assignment" forms a stable matching.

5. Form a project team (ideally with a group of 2-3 students). Please list your teammates and write a few paragraphs about the topic you plan to work on.

[1] See also Echenique/Immorlica/Vazirani Chapter 1, Section 5.1